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Advances in Mathematics 181 (2004) 160–164

ADVANCES IN Mathematics

http://www.elsevier.com/locate/aim

Deligne's topological central extension is universal☆

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Received 7 August 2002; accepted 9 September 2002

Communicated by Johan De Jong

Abstract

We give a purely "local" proof of the fact that the topological central extension of $G(k)$, G an absolutely almost simple algebraic group defined and isotropic over a nonarchimedean local field k, by the finite group $\mu(k)$ of roots of unity in k, constructed by Pierre Deligne, is a universal topological central extension of $G(k)$.

Let k be a local (i.e. nondiscrete locally compact) field different from the field of complex numbers. (Then $k = \mathbb{R}$ or $\mathbb{F}_q((t))$ or it is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers.) Let G be an absolutely (almost) simple, simply connected algebraic group defined over k. In [1, Section 5] (see also [1, 2.9]), Deligne has constructed a topological central extension

$$(*) \quad 1 \to \mu(k) \to G(k)^{\sim} \to G(k) \to 1,$$

of $G(k)$ by the finite cyclic group $\mu(k)$ of roots of unity contained in k. In [5, Section 8], it was shown that, if G is k-isotropic, $(*)$ is a universal topological central extension of $G(k)$. The proof used a global argument, and the main theorem of [3] which asserts that the order of the continuous cohomology group $H^2_c(G(k), \mathbb{R}/\mathbb{Z})$ is at most $\#\mu(k)$. It was also observed in [5], see 8.4–5, that if $k = \mathbb{R}$ and $G(\mathbb{R})$ is not simply connected (then, as is known, see [6], $\pi_1(G(\mathbb{R}))$ is either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$), $(*)$ is the unique 2-sheeted covering of $G(\mathbb{R})$. The purpose of this note is to give purely "local" proofs of these results.

☆Supported in part by a grant from NSF.
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0001-8708/$ - see front matter
doi:10.1016/S0001-8708(03)00048-3

We assume in the sequel that G is k-isotropic.

1. We begin by recalling that if G is split over k, Deligne's central extension $(*)$ is the extension constructed by Matsumoto for the inverse of the Galois symbol [1, Proposition 3.7]. Hence, in this case, the extension $(*)$ is a universal topological central extension of $G(k)$ if k is nonarchimedean, or if $k = \mathbb{R}$ and G is not of type C in which case $\pi_1(G(\mathbb{R})) = \mathbb{Z}$ and $(*)$ is the unique 2-sheeted covering of $G(\mathbb{R})$. Moreover, if k is nonarchimedean, then as an element of $H_c^2(G(k), \mu(k))$ the topological central extension $(*)$ is of order $\#\mu(k)$.

2. Deligne's central extension $(*)$ is functorial in the following sense:

Let $\mathscr{G}$ be an absolutely simple simply connected algebraic k-group and $\varphi : \mathscr{G} \to G$ be a k-homomorphism such that the image of a short coroot of $\mathscr{G}$ has squared length $r = r(\varphi)$; the computation of r is carried out over a field extension of k which splits both $\mathscr{G}$ and G, and the length of coweights of G (i.e., 1-parameter subgroups $\lambda : \mathrm{GL}_1 \to G$) is determined by the Weyl group-invariant integer-valued positive definite inner product on $X_*(T)$, where T is a maximal torus of G and $X_*(T)$ is the group of its 1-parameter subgroups, such that the short coroots have length one. Then the pull-back of $(*)$ under φ is r-times Deligne's central extension of $\mathscr{G}(k)$. This, in particular, implies that if $\mathscr{G} = G$ and φ is a k-rational automorphism of G, then, as r is clearly one, the extension $(*)$ is invariant under φ. Thus *Deligne's central extension* $(*)$ *of* $G(k)$ *is invariant under the group* $(\mathrm{Aut}\, G)(k)$ *of k-rational automorphisms of* G.

3. We consider first the case where G is quasi-split over k. Let S be a maximal k-split torus of G and T be its centralizer; the latter is a maximal torus of G defined over k. It is well known, and easy to see, that for any long root a of the root system of G with respect to S, the root subgroups U_a and U_{-a} are one-dimensional and they together generate a k-subgroup G_a which is normalized by T and which is k-isomorphic to SL_2. Moreover, the unique root of G with respect to T which restricts to a is a long root of the root system of G with respect to T. Now we fix a long root a (of G with respect to S). Let $\mathscr{G} = G_a$ and φ be the natural inclusion of $\mathscr{G}$ in G. Then, clearly, $r = r(\varphi) = 1$, and so the restriction of the extension $(*)$ to $\mathscr{G}(k)$ is Deligne's central extension of $\mathscr{G}(k) \cong \mathrm{SL}_2(k)$. Hence (see **1**), if $k = \mathbb{R}$, the extension $(*)$ is nontrivial and so it is the unique 2-sheeted covering of $G(\mathbb{R})$, and if k is nonarchimedean (and G is quasi-split over k), then as an element of $H_c^2(G(k), \mu(k))$ the topological central extension $(*)$ is of order at least $\#\mu(k)$.

Unless explicitly stated otherwise, in the following k is a nonarchimedean local field.

4. We now assume that G is *not* quasi-split over k. To treat this case, we will use a "nice" simple, simply connected, quasi-split subgroup constructed in [3, 8.1]. For the convenience of the reader we will recall here the description of this subgroup. Let K be the maximal unramified extension of k and $\Gamma = \mathrm{Gal}(K/k)$. In the construction of the affine root system of G, we will use the Γ-invariant valuation v of K such that $v(K^\times) = \mathbb{Z}$. Let T be a maximal K-split torus of G defined over k and containing a maximal k-split torus of G. (According to the Bruhat–Tits theory such a torus exists.) Let Z be the centralizer of T in G. As G is quasi-split over K, Z is a maximal torus; it is defined over k since T is. Let I be an Iwahori subgroup of $G(K)$ defined over k, i.e. which is stable under Γ, such that $I \cap T(K)$ is the maximal bounded subgroup of $T(K)$.

Let Ψ be the affine root system of G with respect to T and Δ be the basis of Ψ determined by the Iwahori subgroup I. As I is defined over k, Δ is stable under the natural action of Γ on Ψ. For an affine root α, let $\dot{\alpha}$ denote its gradient; $\dot{\alpha}$ is a root of G with respect to the maximal K-split torus T. Let $U_{\dot{\alpha}}$ be the corresponding root subgroup; it is a connected unipotent K-subgroup (of dimension $\leqslant 3$) normalized by Z. Let Ω be the Γ-orbit of a *special* root in Δ such that the diagram of $\Delta - \Omega$ is connected except in the case where G is an *inner* k-form of a split group of type A. Let G^* be the algebraic subgroup of G generated by the root subgroups $U_{\pm\dot{\alpha}}$, $\alpha \in \Delta - \Omega$. As $\Delta - \Omega$ is stable under Γ, G^* is defined over k; it is clearly normalized by the maximal k-torus Z. It is shown in [3, 8.1] that G^* is simple, simply connected and quasi-split over k. If G is not an inner k-form of a split group of type A, then G^* is absolutely simple. On the other hand, if G is an inner k-form of a split group of type A, then there is a finite-dimensional central division algebra D/k, and an integer $n \geqslant 2$, such that $G = \mathrm{SL}_{n,D}$ and G^* is the subgroup $\mathrm{SL}_{n,\mathscr{K}}$; where $\mathscr{K}$ is the unramified extension of k of degree equal to the degree of D, $\mathscr{K}$ thought of as a maximal subfield of D in terms of a fixed embedding; thus in this case $G^* \cong R_{\mathscr{K}/k}(\mathrm{SL}_n)$.

As the maximal k-torus Z of G normalizes the subgroup G^*, the root system of G^* with respect to Z is a subroot system of the root system of G. Using the description of G^* given above and the "local indices" given in [7, pp. 62–66], we can see that except where G is a k-form of a split group of type C, the root system of G^* (with respect to Z) contains a long root of the root system of G. Thus, if G is not an inner k-form of a split group of type either A or C, for $\mathscr{G} = G^*$ and φ the inclusion $\mathscr{G} \hookrightarrow G$, $r = r(\varphi) = 1$, and hence the restriction of the central extension $(*)$ to $G^*(k)$ is Deligne's central extension of $G^*(k)$. Since G^* is quasi-split over k, this, combined with the observation in **3**, implies that (if G is not an inner k-form of a split group of type either A or C, then) as an element of $H_c^2(G(k), \mu(k))$ the topological central extension $(*)$ is of order at least $\#\mu(k)$.

5. We will now consider the case where G is a k-isotropic inner k-form of a split group of type A and let D, $\mathscr{K}$ and n be as in **4**. Then $G^* = \mathrm{SL}_{n,\mathscr{K}}$. As Deligne explained to us (see [1, 3.9(iii) and Proposition 5.5(i)]), the restriction of the central extension $(*)$ of $G(k) = \mathrm{SL}_n(D)$ to the subgroup $G^*(k) = \mathrm{SL}_n(\mathscr{K})$ is the central extension of $\mathrm{SL}_n(\mathscr{K})$ by $\mu(k)$ obtained from Deligne's extension of $\mathrm{SL}_n(\mathscr{K})$ by $\mu(\mathscr{K})$ using the map $x \mapsto x^m$ of $\mu(\mathscr{K})$ onto $\mu(k)$, where $m = \#\mu(\mathscr{K})/\#\mu(k)$. Therefore, from the observation in **1**, we conclude that as an element of $H_c^2(G(k), \mu(k))$ the topological central extension $(*)$ is of order at least $\#\mu(k)$.

6. We will now treat the case where G is a nonsplit k-form of a split group of type C. Let T, Z, Δ and Ω be as in **4**. Then $Z = T$, Ω consists of exactly two roots, say ω and ω', and these roots are long and orthogonal to each other. Let $G^\bullet$ be the subgroup of G generated by the root subgroups $U_{\pm\dot{\omega}}$ and $U_{\pm\dot{\omega}'}$. It has been observed in [3, 7.36], that $G^\bullet$ is a simply connected k-subgroup of G normalized by T and it is k-isomorphic to $R_{\mathscr{K}/k}(\mathrm{SL}_2)$; where $\mathscr{K}$ is the unramified quadratic extension of k.

The root system of $G^{\bullet}$ with respect to T consists of the (long) roots $\pm\dot{\omega}$, $\pm\dot{\omega}'$. Now as in **5**, we conclude that as an element of $H_c^2(G(k), \mu(k))$ the topological central extension $(*)$ is of order at least $\#\mu(k)$.

7. Since G was assumed to be isotropic over k, $G(k)$ is perfect, i.e. it equals its commutator subgroup. From this it follows that for any given embedding of $\mu(k)$ in $\mathbb{R}/\mathbb{Z}$, the induced homomorphism

$$H_c^2(G(k), \mu(k)) \to H_c^2(G(k), \mathbb{R}/\mathbb{Z})$$

is injective. Now as Deligne's topological central extension of $G(k)$ by $\mu(k)$, as an element of $H_c^2(G(k), \mu(k))$, is of order at least $\#\mu(k)$, the continuous cohomology group $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$ is of order at least $\#\mu(k)$. On the other hand, we know from the main theorem of [3] that $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$ is isomorphic to a subgroup of the Pontrjagin dual $\hat{\mu}(k) \coloneqq \mathrm{Hom}(\mu(k), \mathbb{R}/\mathbb{Z})$ of $\mu(k)$. Therefore,

$$H_c^2(G(k), \mu(k)) \cong \hat{\mu}(k) \cong H_c^2(G(k), \mathbb{R}/\mathbb{Z}),$$

and hence $G(k)$ admits a universal topological central extension (see [3, Section 10; 2, Theorem 12]; note that according to a theorem of [8] the cohomology groups of $G(k)$ defined in terms of measurable cochains are isomorphic to the continuous cohomolgy groups). Moreover, the "topological fundamental group" of $G(k)$ is isomorphic to the Pontrjagin dual $(= \mu(k))$ of $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$, see [2, Proposition 5], so *Deligne's topological central extension* $(*)$, *of* $G(k)$ *by* $\mu(k)$, *which as an element of* $H_c^2(G(k), \mu(k))$ *is of order precisely* $\#\mu(k)$, *is a universal topological central extension of* $G(k)$.

Since Deligne's central extension generates $H_c^2(G(k), \mu(k))$ and it is invariant under the group $(\mathrm{Aut}\, G)(k)$, see **2** above, we conclude that $(\mathrm{Aut}\, G)(k)$ *operates trivially on* $H_c^2(G(k), \mu(k))$, *and so also on* $H_c^2(G(k), \mathbb{R}/\mathbb{Z})$.

8. If $\mathscr{G}$ is an absolutely simple simply connected algebraic group defined and anisotropic over k, then there exists a finite-dimensional central division algebra D/k such that $\mathscr{G}(k) = \mathrm{SL}_1(D)$. It is easy to see that any k-rational automorphism of $\mathscr{G}$ equals the conjugation action of an element of $D^\times$ on $\mathscr{G}$. It is known [4, Theorem 4.6] that *if either the degree* d *of* D *or the characteristic* p *of the residue field of* k *is different from* 2, *then the induced action of* $D^\times$ *on* $H_c^2(\mathrm{SL}_1(D), \mathbb{R}/\mathbb{Z})$ *is trivial.*

Let J be the p-primary component of $\mathbb{Q}/\mathbb{Z}$ and J' be the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of 0 and all elements of order prime to p. Then $\mathbb{Q}/\mathbb{Z} = J \oplus J'$. We will view J, J' and $\mathbb{Q}/\mathbb{Z}$ as topological groups with discrete topology and think of $\mu(k)$ as a subgroup of $\mathbb{Q}/\mathbb{Z}$ in terms of a fixed embedding. It is known that $H_c^2(\mathrm{SL}_1(D), J')$ is trivial [4, Proposition 2.1]. Therefore, $H_c^2(\mathrm{SL}_1(D), J) \hookrightarrow H_c^2(\mathrm{SL}_1(D), \mathbb{Q}/\mathbb{Z})$ is an isomorphism and hence Deligne's topological central extension of $\mathrm{SL}_1(D)$ by $\mu(k)(\hookrightarrow \mathbb{Q}/\mathbb{Z})$ gives us an element $\mathfrak{c}$ of $H_c^2(\mathrm{SL}_1(D), J)$. We expect that $\mathfrak{c}$ is of order $\#\mu(k)_{\mathrm{wild}}$. We can show this to be the case if either d or p is odd, or if $p = 2$, $d \neq 2$, and there is a quadratic cyclotomic extension F of k such that $N_{F/k}(\mu(F)) = \mu(k)$ (for example, if k contains a primitive fourth root of unity), using the fact that

Deligne's central extension of $SL_1(D)$ is the restriction of Deligne's central extension of $SL_n(D)$, $n \geqslant 2$, which follows from the functoriality of the extensions described in **2**, and a commutator computation as in the proof of Theorem 8.2 in [4]; see also [5, 8.5].

9. Now let $k = \mathbb{R}$. If $G(\mathbb{R})$ is simply connected as a topological space, then it does not admit any nontrivial coverings and hence the topological central extension $(*)$ of $G(\mathbb{R})$ by $\mu(\mathbb{R}) = \{\pm 1\}$ is necessarily trivial. If $G(\mathbb{R})$ is not simply connected, then as G is an absolutely simple simply connected algebraic group, the fundamental group of $G(\mathbb{R})$ is either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$; see [6]. It is an empirical fact, which can be checked using the tables in [6], that if the fundamental group of $G(\mathbb{R})$ is nontrivial, then for any long root a in the root system of G with respect to a maximal $\mathbb{R}$-split torus S, the root subgroups U_a and U_{-a} are one-dimensional and the subgroup G_a which they generate together is $\mathbb{R}$-isomorphic to SL_2. It is clear that G_a is normalized by the centralizer Z of S in G. Now *whenever* (for some, and hence all, long root a) G_a *is isomorphic to* SL_2 (or, equivalently, U_a is one-dimensional) arguing as in **3**, we conclude that $G(\mathbb{R})$ is not simply connected and *Deligne's extension* $(*)$ *of* $G(\mathbb{R})$ by $\mu(\mathbb{R})$ *is the unique nontrivial* 2*-sheeted covering of* $G(\mathbb{R})$.

Acknowledgments

The author is grateful to Pierre Deligne for his comments.

References

[1] P. Deligne, Extensions centrales de groupes algébriques simplement connexes et cohomologie Galoisienne, Publ. Math. IHES 84 (1996) 35–89.
[2] C.C. Moore, Group extensions and cohomology for locally compact groups, IV, Trans. Amer. Math. Soc. 221 (1976) 35–58.
[3] G. Prasad, M.S. Raghunathan, Topological central extensions of semi-simple groups over local fields, Ann. Math. 119 (1984) 143–268.
[4] G. Prasad, M.S. Raghunathan, Topological central extensions of $SL_1(D)$, Invent. Math. 92 (1988) 645–689.
[5] G. Prasad, A.S. Rapinchuk, Computation of the metaplectic kernel, Publ. Math. IHES 84 (1996) 91–187.
[6] J. Tits, Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen, Springer, Lecture Notes in Mathematics, Springer, Berlin, New York, 1967.
[7] J. Tits, Reductive groups over local fields, Proc. Symp. Pure Math. Amer. Math. Soc. Part 1 33 (1979) 29–69.
[8] D. Wigner, Algebraic cohomology of topological groups, Trans. Amer. Math. Soc. 178 (1973) 83–93.